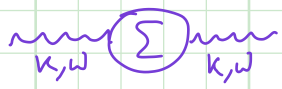
Plan : (1) Resummation (RPA)

(2) Imaginary freq. poles + spontaneous symmetry breaking

Last time : Diagrammatically we can represent the fermion bubble as:

 $= \tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\, \tilde{G}^{ph(0)}_K(\omega)$

photon → bubble → photon

$\Sigma_K(\omega)$ is called the photon self energy

<u>Resummation of diagrams:</u>

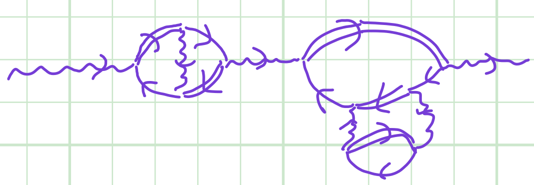
we can improve on the photon propagator without much additional work using diagram resummation.

consider the following series of diagrams that make up the photon propagator:

$\tilde{G}^{photon}_K(\omega) =$ 

$$= \tilde{G}^{ph(0)}_K(\omega) + \tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\, \tilde{G}^{ph(0)}_K(\omega) + \tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\, \tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\, \tilde{G}^{ph(0)}_K(\omega) + \dots$$

Note 1 : This series is, to be sure, missing some terms ⇒ examples?

 etc.

Note 2 : This series is easy to sum using a simple "trick"

(a) consider the power series in small $\Sigma_K(\omega)$:

$$\frac{\tilde{G}^{ph(0)}_K(\omega)}{1 - \tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)} = \tilde{G}^{ph(0)}_K(\omega) + \left[\tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\right] \tilde{G}^{ph(0)}_K(\omega) + \left[\tilde{G}^{ph(0)}_K(\omega)\, \Sigma_K(\omega)\right]^2 \tilde{G}^{ph(0)}_K(\omega) + \dots$$

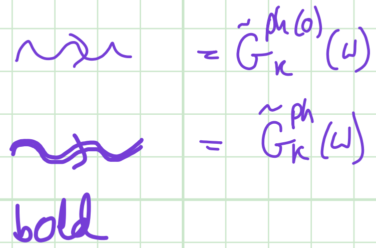
$= \tilde{G}^{ph}_K(\omega)$ [Exactly the answer that we were looking for]

(b) What is the meaning of the above expression? multiply both sides by the denominator :

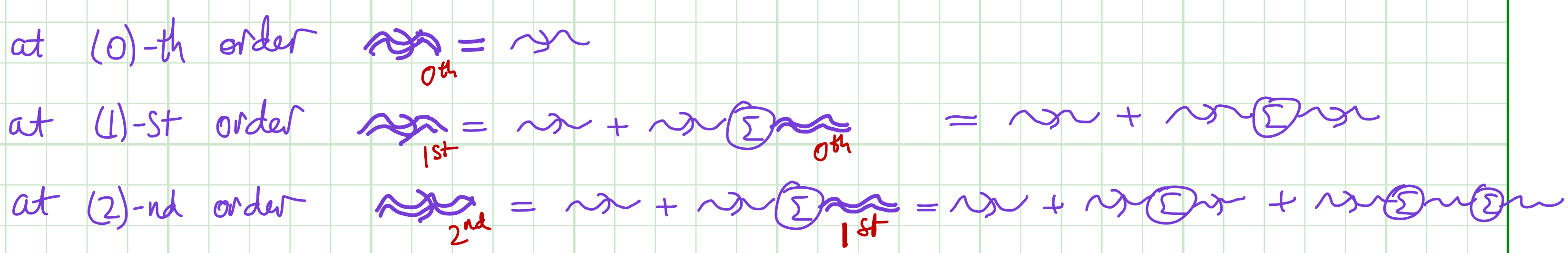
$$\left(1-\tilde{G}_K^{ph(0)}(\omega)\Sigma_K(\omega)\right)\tilde{G}_K^{ph}(\omega)=\tilde{G}_K^{ph(0)}(\omega)$$

rearrange $\Rightarrow$ $\tilde{G}_K^{ph}(\omega)=\tilde{G}_K^{ph(0)}(\omega)+\tilde{G}_K^{ph(0)}(\omega)\,\Sigma_K(\omega)\,\tilde{G}_K^{ph}(\omega)$ [Dyson eq.]

This equation constructs the full photon Green function iteratively. Let's Demonstrate diagrammatically.



Using diagrams the Dyson equation looks like this:



etc ...

series generated recursively.

Note 3: How is the photon propagator affected by semiconductor?

let's plug $\tilde{G}_K^{ph(0)}(\omega)=\dfrac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}$ into the Dyson eq:

$$\tilde{G}_K^{ph}(\omega)=\frac{\tilde{G}_K^{ph(0)}(\omega)}{1-\tilde{G}_K^{ph(0)}(\omega)\Sigma_K(\omega)}=\frac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}\left[1-\frac{2\omega_K}{\omega^2-\omega_K^2+i\varepsilon}\Sigma_K(\omega)\right]^{-1}$$

$$=\frac{2\omega_K}{\cancel{\omega^2-\omega_K^2+i\varepsilon}}\left[\frac{\omega^2-\omega_K^2+i\varepsilon-2\omega_K\Sigma_K(\omega)}{\cancel{\omega^2-\omega_K^2+i\varepsilon}}\right]^{-1}=\frac{2\omega_K}{\omega^2-\omega_K^2-2\omega_K\Sigma_K(\omega)+i\varepsilon}$$

$\Rightarrow$ It looks like $\Sigma_K(\omega)$ is shifting the energy, ω_K, of the photon.
Hence $\Sigma_K(\omega)$ is called the self-energy.

Note 4: What is the relation between $\Sigma_K(\omega)$ and quantities we know like χ?

Conventionally in optics, if we want photon propagation in medium, we change the speed of light: $\omega_K=cK\rightarrow\tilde{\omega}_K=\tilde{c}K$. Let's compare photon propagating with alternate speed of light to the above answer:

In medium:

$$\tilde{G}^{ph}_k(\omega) = \frac{2\tilde{\omega}_k}{\omega^2 - \tilde{\omega}_k^2 + i\varepsilon} \Rightarrow \text{pole located at: } \omega^2 = \tilde{\omega}_k^2 = \tilde{c}^2k^2 = \frac{c^2k^2}{n^2}$$

Our calculation, pole located at: $\omega^2 = \omega_k^2 + 2\omega_k\Sigma_k = c^2k^2 + 2ck\Sigma_k(\omega)$

Comparing the two: $\frac{c^2k^2}{n^2} = c^2k^2 + 2ck\Sigma_k(\omega) \Rightarrow c^{\cancel{2}}k^{\cancel{2}}(1-n^2) = 2\cancel{ck}\,\Sigma_k(\omega)$

$$\boxed{(1-n^2) = -\chi = \frac{2\Sigma_k(\omega)}{ck}}$$

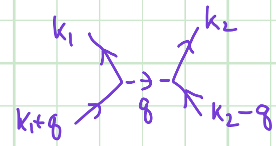
Pair and spin susceptibilities in Fermi Gas.

Consider a Fermi gas with point contact interactions

$$H = \sum_{k\sigma} \underbrace{\left(\frac{k^2}{2m} - \mu\right)}_{\xi_k} c^\dagger_{k\sigma} c_{k\sigma} + U\int dx_1\, dx_2\; c^\dagger_{x_1\uparrow} c_{x_1\uparrow}\; c^\dagger_{x_2\downarrow} c_{x_2\downarrow}\; \delta(x_1 - x_2)$$

$$\Downarrow$$

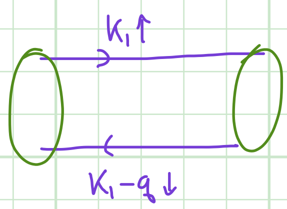
$$U\sum_{k_1k_2q} c^\dagger_{k_1\uparrow} c_{k_1+q\uparrow}\; c^\dagger_{k_2\downarrow} c_{k_2-q\downarrow}$$

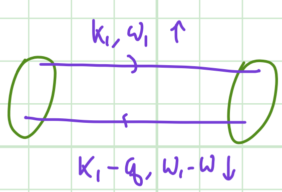


Let us compute the spin susceptibility

$$\chi(q,\omega) = \int dt\, e^{i\omega t}\langle S^+(q,t)\, S^-(-q,0)\rangle = \int dt\, e^{i\omega t}\left\langle \left(\sum_{k_1} c^\dagger_{k_1\uparrow}(t)\, c_{k_1-q\downarrow}(t)\right)\left(\sum_{k_2} c^\dagger_{k_2\downarrow}(0)\, c_{k_2+q\uparrow}(0)\right)\right\rangle$$

The non interacting part — 0^{th} order — $(-i)(-1) \leftarrow$ 1 fermion loop } or something...

$\chi^{(0)}(q,t) = S^+(q,t)$  $S^-(-q,0) = +i\int dk_1\; G^{(0)}_{k_1\uparrow}(t)\, G^{(0)}_{k_1-q\downarrow}(-t)$

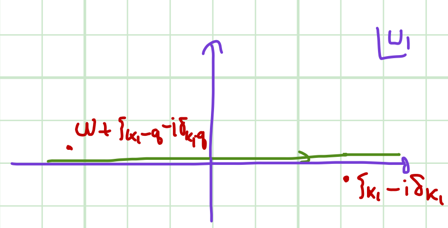
$\chi^{(0)}(q,\omega) = S^+(q,\omega)$  $S^-(-q,-\omega) = +i\int dk_1 \frac{d\omega_1}{(2\pi)}\; G^{(0)}_{k_1\uparrow}(\omega_1)\, G^{(0)}_{k_1-q\downarrow}(\omega_1-\omega)$

$$= \int dk_1 \frac{d\omega_1}{2\pi}\left[\frac{1}{\omega_1 - \xi_{k_1} + i\delta_{k_1}}\;\frac{1}{\omega_1 - \omega - \xi_{k_1-q} + i\delta_{k_1-q}}\right] \qquad \delta_k = \delta\,\mathrm{sign}(\xi_k)$$

Strategy: Perform the $d\omega_1$ integral first, then perform the dk_1 integral

poles: $\omega_1 = \xi_{k_1} - i\delta_{k_1}$

$\omega_1 = \omega + \xi_{k_1-q} - i\delta_{k_1-q}$



contour integral non-zero only if one pole in upper-half plane + the other in lower half-plane